Exponentially Convergent Multiscale Methods

for solving elliptic and Helmholtz's equation

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Computing and Mathematical Sciences, Caltech

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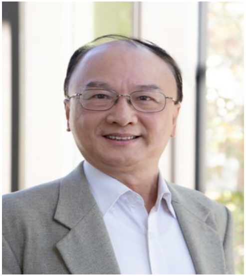
Collaborators and Related Papers **[Chen, Hou, Wang 2021,2021, 2022]**



Thomas Hou
Caltech

Yixuan Wang
Caltech

- Exponential Convergence for Multiscale Linear Elliptic PDEs via Adaptive Edge Basis Functions, SIAM-MMS, 2021
- Exponentially Convergent Multiscale Methods for 2D High Frequency Heterogeneous Helmholtz Equations, SIAM-MMS, 2022
- Exponentially Convergent Multiscale Finite Element Method, CAMC, 2022 (review paper)

Outline

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Multiscale Problems: Heterogeneous Media and High Frequency Waves

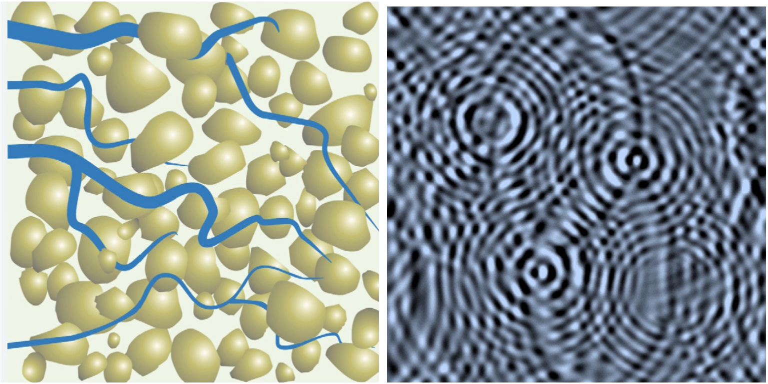

Figure: Heterogeneity and high frequency

Figure credited to Google online search

Mathematical Setup

Model problem:

$$-\nabla \cdot (A\nabla u) + Vu = f, \text{ in } \Omega, \quad \text{w/ boundary conditions}$$

(subsurface flows, diffusions, elasticity, waves)

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$$-\nabla \cdot (A \nabla u) + V u = f, \text{ in } \Omega, \quad \text{w/ boundary conditions}$$

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Mathematical conditions:

- Heterogeneity:

$$A, V \in L^{\infty}(\Omega), \text{ and } 0 < A_{\min} \leq A(x) \leq A_{\max} < \infty$$

- High frequency:

e.g., $V = -k^2$ (Helmholtz's equation)

Outline

Literature: Scale Separation v.s. Continuum of Scales

Explicit scale parameter ϵ: $A = A(x, x/\epsilon)$

- Theory: with scale separation and periodicity assumptions
  $\Rightarrow$ there is a homogenized $A_0 = A_0(x)$ when $\epsilon \to 0$
- Coarse scale behaviors of the solution are often identified using asymptotic analysis and homogenization theory

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Continuum of scales: $A \in L^\infty$

- Find local basis functions that capture the fine scale information, and use them to identify the correct coarse scale behaviors of the solution
- "Coarse scale" becomes a design choice in numerical approximation

Galerkin's method:

- Construct basis functions and plug them into the variational form
- Key: Quasi-optimality, i.e.,

Galerkin solution err $\sim$ optimal approx-err in $\|\cdot\|_{\mathcal{H}(\Omega)}$

$\mathcal{H}(\Omega)$ is the energy norm

Numerical Approximation for Multiscale PDEs with Continuum Scales

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Challenges:

- Heterogeneity $\Rightarrow$ u or ∇u is oscillatory
  (!) approx-err of FEM can be arbitrarily bad [Babuška, Osborn 2000]

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  e.g., $\|u\|_{\mathcal{H}(\Omega)} \le C_{\mathsf{stab}}(k)\|f\|_{L^2(\Omega)}$ for $C_{\mathsf{stab}}(k) \succeq 1 + k^\gamma$

  (!) approx-err amplified by the stability constant
  - quasi-optimality also deteriorates, e.g., require $H = O(1/k^2)$
  - phenomenon known as pollution effect [Babuška, Sauter, 1997]

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(!) Need high accuracy approximation

$\mathcal{H}(\Omega)$ is the energy norm

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If the function is non-smooth?

Warm-up: A-harmonic functions

- Function space: assume $0 < A_{\min} \le A(x) \le A_{\max} < \infty$

$$U(D) := \{v \in H^1(D), -\nabla \cdot (A\nabla v) = 0 \text{ in } D\}/\mathbb{R}$$

- Norm:

$$\|v\|_{H^1_A(D)} := \|A^{1/2}\nabla v\|_{L^2(D)}$$

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Theorem [Babuška, Lipton 2011]

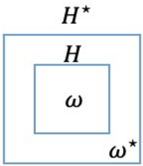
The singular values $\sigma_m(R)$ of the restriction operator

$$R : (U(\omega^*), \|\cdot\|_{H^1_A(\omega^*)}) \to (U(\omega), \|\cdot\|_{H^1_A(\omega)})$$

decays nearly exponentially fast:

$$\sigma_m(R) \leq C_\epsilon \exp\left(-m^{\frac{1}{d+1}-\epsilon}\right)$$

for some C_ϵ independent of H and m



Approximating A-harmonic functions can be exponentially convergent

- For any $u \in H^1_A(\omega^*)$, there are m functions $v_j, 1 \leq j \leq m$

$$\|u - \sum_{j=1}^{m} c_j v_j\|_{H^1_A(\omega)} \leq C_\epsilon \exp\Big(-m^{\frac{1}{d+1}-\epsilon}\Big)\|u\|_{H^1_A(\omega^*)}$$

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Summarize the property: Restrictions of A-harmonic functions are of low approximation complexity

Multiscale Spectral Generalized Finite Element Method (MS-GFEM)

[Babuška, Lipton 2011], [Babuška, Lipton, Sinz, Stuebner 2020], [Ma, Scheichl, Dodwell 2021]

For elliptic equations with rough coefficients

$$-\nabla \cdot (A\nabla u) = f, \text{ in } \Omega, \quad \text{w/ boundary conditions}$$

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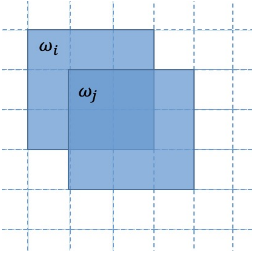
Decompose the solution to local components:

$$u = \sum_i \eta_i u = \sum_i \eta_i u^{\mathsf{h}}_{\omega_i} + \sum_i \eta_i u^{\mathsf{b}}_{\omega_i}$$

- Overlapped domain decomp.

  ω_i is of size $O(H)$



- Partition of unity functions
  $\sum_i \eta_i = 1$, supp $\eta_i = \omega_i$, η_i smooth
- Harmonic-bubble splitting

$$\begin{cases} -\nabla \cdot (A\nabla u^{\mathsf{h}}_{\omega_i}) = 0, & \text{in } \omega_i \\ u^{\mathsf{h}}_{\omega_i} = u, & \text{on } \partial\omega_i \end{cases}$$

$$\begin{cases} -\nabla \cdot (A\nabla u^{\mathsf{b}}_{\omega_i}) = f, & \text{in } \omega_i \\ u^{\mathsf{b}}_{\omega_i} = 0, & \text{on } \partial\omega_i \end{cases}$$

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$$\Rightarrow u = \underbrace{\sum_i \sum_{k=1}^{m} c_{i,k} v_{i,k}}_{\text{(I)}} + O\left(\exp\left(-m^{\frac{1}{d+1}-\epsilon}\right)\|u\|_{H^1_A(\Omega)}\right) + \underbrace{\sum_i \eta_i u^{\mathsf{b}}_{\omega_i}}_{\text{(II)}}$$

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Offline: for a given A, one constructs local basis functions

- Compute $v_{i,k}$ in (I) by solving local spectral problems

Online: given any source term f, one solves u

- Compute (II) by solving local equations
- Compute (I) by Galerkin's method with offline basis functions

Note: Local problems are solved on a fine mesh $h < H$

Outline

Our Contributions: New Local Structures and Global Coupling

Multiscale methods for Helmholtz's equation

[Peterseim 2017], [Peterseim, Verfürth 2020], [Fu, Li, Craster, Guenneau 2021]

$$-\nabla \cdot (A\nabla u) - k^2 u = f, \text{ in } \Omega, \quad \text{w/ boundary conditions}$$

- No pre-existing results on achieving exponential convergence
- Question: What is the structure of low approximation complexity?
- Difficulty: Indefiniteness of the operator

Non-overlapped domain decomposition [Hou, Wu 1997], [Hou, Liu 2015]

- Advantage: More localized domains, local functions less overlapped (more efficient in stiffness matrix assembly)
- Difficulty: Non-overlapping, interaction through edges

Helmholtz-harmonic functions

- Function space:

$$U_k(D) := \{v \in H^1(D), -\nabla \cdot (A\nabla v) - k^2 v = 0 \text{ in } D\}/\mathbb{R}$$

- Energy norm:

$$\|v\|^2_{\mathcal{H}(D)} := \|A^{1/2}\nabla v\|^2_{L^2(D)} + \|kv\|^2_{L^2(D)}$$

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Theorem [Chen, Hou, Wang 2021], [Ma, Alber, Scheichl 2021]

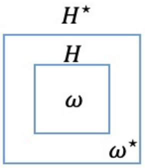
Let $H^\star = O(1/k)$, then singular values $\sigma_m(R)$ of the restriction operator for Helmholtz-harmonic functions

$$R : (U_k(\omega^*), \|\cdot\|_{\mathcal{H}(\omega^*)}) \to (U_k(\omega), \|\cdot\|_{\mathcal{H}(\omega)})$$

decays nearly exponentially fast:

$$\sigma_m(R) \le C_\epsilon \exp\Big(-m^{\frac{1}{d+1}-\epsilon}\Big)$$

for some C_ϵ independent of k, H and m



Key: $H^\star = O(1/k) \Rightarrow$ *the Helmholtz operator is locally positive definite*

Multiscale methods for Helmholtz's equation

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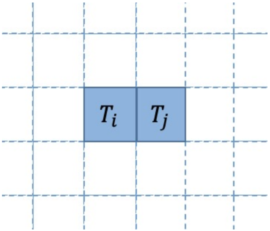

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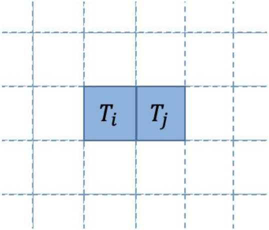
1. Decomposition using indicator funcs

$$u = \sum_i \mathbb{1}_{T_i} u = \sum_i \mathbb{1}_{T_i} u^{\mathsf{h}}_{T_i} + \underbrace{\sum_i \mathbb{1}_{T_i} u^{\mathsf{b}}_{T_i}}_{\text{locally computable}}$$



Non-overlapped Domain Decomposition: Localization and Coupling

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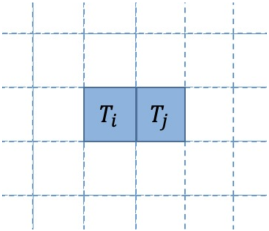
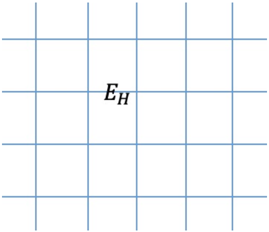


2. Focus on edge functions

$$\sum_i \mathbb{1}_{T_i} u^{\mathsf{h}}_{T_i} = Q\tilde{u}^{\mathsf{h}} \in H^{1/2}(E_H)$$

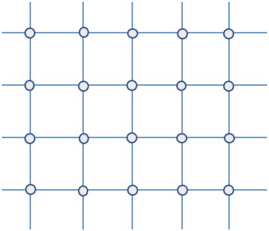
- where $Q : H^{1/2}(E_H) \to H^1(\Omega)$ is the Helmholtz-harmonic extension operator



3. Edge localization

$$\tilde{u}^{\mathsf{h}} = \underbrace{I_H \tilde{u}^{\mathsf{h}}}_{\text{Nodal interp.}} + \underbrace{(\tilde{u}^{\mathsf{h}} - I_H \tilde{u}^{\mathsf{h}})}_{\text{Decoupled to each edges}}$$

- $I_H \tilde{u}^{\mathsf{h}} = \sum_n u(x_n)\psi_n$ spanned by nodal basis funcs



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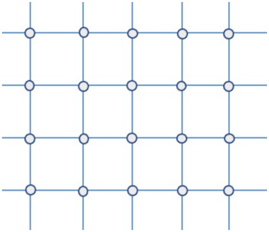
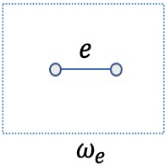


4. Oversampling for exp. accuracy

$$(\tilde{u}^{\mathsf{h}} - I_H \tilde{u}^{\mathsf{h}})|_e$$

$$= \sum_{j=1}^{m} c_{j,e}\tilde{v}_{j,e} + O\left(\exp\left(-m^{\frac{1}{d+1}-\epsilon}\right)\right)$$

$$+ \underbrace{(u^{\mathsf{b}}_{\omega_e} - I_H u^{\mathsf{b}}_{\omega_e})|_e}_{\text{small and locally computable}}$$



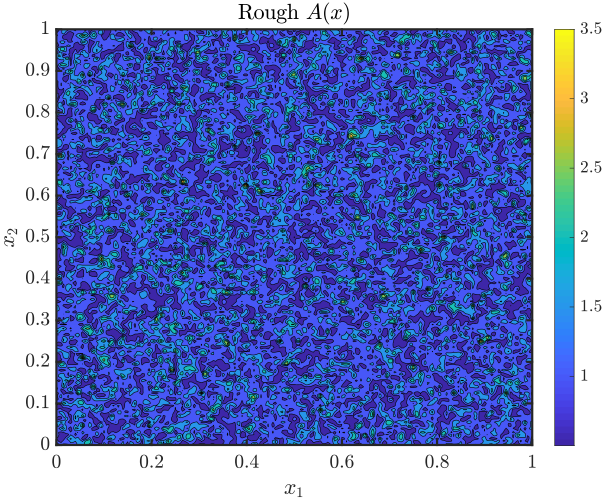
Theorem [Chen, Hou, Wang 2021]

The following holds for the solution u of Helmholtz's equation

$$u = \left(\sum_n b_n \psi_n + \sum_e \sum_{j=1}^{m} c_{j,e} v_{j,e} \right) + \left(\sum_i \mathbb{1}_{T_i} u^{\mathsf{b}}_{T_i} + \sum_e Q(u^{\mathsf{b}}_{\omega_e} - I_H u^{\mathsf{b}}_{\omega_e})|_e \right) + O\left(\exp\left(-m^{\frac{1}{d+1}-\epsilon}\right) (\|u\|_{\mathcal{H}(\Omega)} + \|f\|_{L^2(\Omega)}) \right)$$

The problem set-up

- equation
$$-\nabla \cdot (A\nabla u) + Vu = f, \text{ in } \Omega = [0,1]^2$$
- boundary condition: mixed (Dirichlet + Neumann + Robin)
- $A(x) = |\xi(x)| + 0.5$ where $\xi(x)$ is piecewise linear functions with values as unit Gaussians r.v.; piecewise scale: 2^{-7}
- $-V/k^2$ draws from the same random field; $k = 2^5$
- $f(x_1, x_2) = x_1^4 - x_2^3 + 1$


Rough $A(x)$
x_2
x_1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
3.5
3
2.5
2
1.5
1

The mesh

- quadrilateral mesh
- fine mesh size $h = 2^{-10}$, coarse mesh size $H = 2^{-5}$

The accuracy of ExpMsFEM's solution compared to fine mesh solution

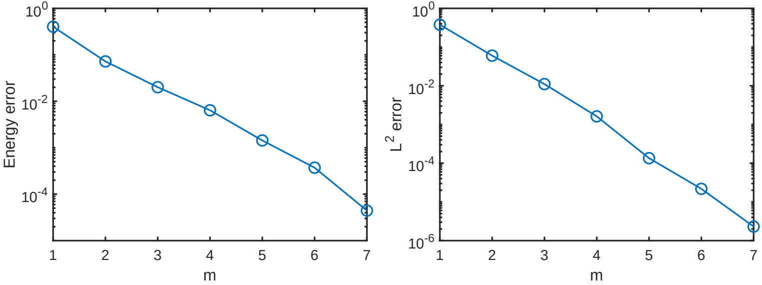


Figure: Numerical results for the mixed boundary and rough field example. Left: $e_{\mathcal{H}}$ versus m; right: e_{L^2} versus m. Number of basis functions $(2m+1)/H^2$